A HYDRODYNAMIC APPROACH TO THE BEHAVIOR OF AN ANTENNA IN A WARM PLASMA

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ABSTRACT

The behavior of an antenna in a warm plasma is determined through the solution of the linearized hydrodynamic equations and Maxwell's equations, using an approximate ion sheath boundary condition. The latter consists in the represention of the support of the sheath by a static electric double layer of finite potential, resulting in a model which has an abrupt discontinuity in electron concentration at the sheath edge. This approximation represents the static characteristics of an actual sheath, and is a reasonable approximation for r.f. fields with wavelengths much greater than the actual thickness of the sheath edge. As such wavelengths are also necessary to the neglect of Landau damping the ranges of validity of the hydrodynamic approach and of the present sheath model are identical. The presence of a time varying (r.f.) electron condensation at the sheath boundary is shown to modify the electron direct current through the sheath, and to introduce a time varying component of this current as well. This leads to theoretical representations of antenna impedance and of the direct current collected by an antenna which are similar to those observed by experimenters.

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1. Introduction

It has been consistently observed, in both ionospheric and laboratory experiments, that predictions of the behavior of antennas in plasmas, based on cold plasma theory, are inadequate to explain various details of the measurements, particularly in those of the real part of either the admittance or impedance of an antenna, and as well of the direct current collected by an antenna. Improvements in the theory naturally result when the finite temperature of the electrons of the plasma is taken into account. Ideally this should be implemented through the simultaneous solution of Boltzmann's and Maxwell's equations. However, the mathematical difficulties of this approach are considerable unless approximations are made, one of which is the use of the linearized hydrodynamic equations to represent the implications of Boltzmann's equation [cf. Gould, 1959; Oster, 1960; Cohen, 1961, 1962a, 1962b; Wait, 1964; Fejer, 1964, and Balmain, 1965, 1966].

The purpose of the present paper is to show further application of the hydrodynamic approach in the determination of the behavior of an antenna in a warm plasma. Essential to this is an approximation that the sheath which surrounds an antenna in a plasma may be treated as if it were a sharply defined discontinuity in the electron concentration, as if supported by a static electric double layer of finite potential. Fejer [1964] has employed a similar condition embodied in an assumption of a reflecting boundary, which implies an infinite potential barrier.

Justification for the assumption of discontinuous electron concentration as representative of a sheath may be found in the work of Pavkovich [1964], in which it is shown that most of the change in electron concentration may take place in about one Debye length. This of course places a lower limit on wavelengths for which the present theory is applicable; however, it is also the limit imposed on the hydrodynamic theory by the neglect of Landau damping.

An important implication of the assumed finite potential barrier approximation is that it permits a finite electron current flow to the antenna. The presence of an oscillating (r.f.) field produces an oscillating electron condensation at the sheath boundary. As the current through the sheath depends on the discontinuity in the electron pressure at the barrier, the oscillating part of the electron pressure necessarily produces a time varying component of the electron current through the barrier. This is not unlike the actual situation at a sheath, and it is shown in the following analysis that the assumed boundary condition leads to results which compare favorably with measurements which have been reported in the literature.

2. The Field Equations

Waves in a warm plasma may be characterized by a field Φ, which is the collum matrix

$$\Phi = \begin{vmatrix} E \\ H \\ v \\ p \end{vmatrix} \qquad (1)$$

where E is the electric field intensity (a vector), H is the magnetic field intensity (a vector), v is the mean electron velocity in the plasma (a vector) and p is the electron pressure perturbation (a scalar). The field Φ is produced by a source distribution, denoted Ψ, which, in notation similar to that of Cohen (1962a), is assigned the form

$$\Psi = \begin{vmatrix} J \\ K \\ -F \\ -Q \end{vmatrix} \qquad (2)$$

where J and K are the conventional electric and magnetic current densities, F is a mechanical force density, and Q is the rate of electron creation (or emission) by the source. The relationship between the field Φ and source Ψ is

$$\Psi = M\Phi \qquad (3)$$

where the operator M is a statement of Maxwell's equations and, in this analysis, the linearized hydrodynamic equations for momentum and continuity. If the time dependence of the field and source quantities is assumed to be $e^{i\omega t}$ then M is given by the tensor

$$M = \begin{vmatrix} -i\omega\varepsilon_o & \nabla\wedge & \frac{\delta_o e}{m} & 0 \\ -\nabla\wedge & -i\omega\mu_o & 0 & 0 \\ -\frac{\delta_o e}{m} & 0 & -i\omega\delta_o \mathcal{U} & -\nabla \\ 0 & 0 & -\nabla\cdot & -i\omega\kappa_o \end{vmatrix} \tag{4}$$

where the usual Lorentz gas approximations have been made, and where the notational definitions are

$\frac{e}{m}$ magnitude of charge to mass ratio for an electron

ε_o permittivity of free space

μ_o permeability of free space

δ_o electron mass density (m times the electron concentration)

κ_o inverse elasticity for electrons

$\mathcal{U}$ momentum distribution tensor

(Portions of this notation were suggested to the author by G. A. Deschamps.) The momentum distribution tensor $\mathcal{U}$ may be expressed in general as

$$\mathcal{U} = -\frac{\omega_p^2}{\omega^2}\chi^{-1} \tag{5}$$

where χ is the dielectric susceptibility of a cold plasma as given by the Appleton-Hartree approximation, and ω_p is the plasma frequency

$$\omega_p = \frac{e}{m} \sqrt{\frac{\delta_o}{\varepsilon_o}} \tag{6}$$

For an isotropic plasma (i.e., no static magnetic field) $\mathcal{U}$ becomes the scalar U defined by

$$\mathcal{U} = U = 1 - i\,\nu/\omega \tag{7}$$

where ν is the electron collision frequency for momentum transfer.

The significance of the use of the above notational convention is that ε_o, μ_o, δ_o, and κ_o may be regarded as intrinsic properties of the medium. These quantities are related by

$$\begin{aligned} \varepsilon_o \mu_o &= 1/c^2 \\ \delta_o \kappa_o &= 1/a^2 \end{aligned} \tag{8}$$

where c is the velocity of light in vacuum and a is the so-called sonic velocity for an electron gas, which is defined by

$$a^2 = \frac{\gamma k_B T}{m} \tag{9}$$

where γ is the ratio of specific heats, k_B is Boltzmann's constant, and T is the electron temperature. It should be further noted that just as $\varepsilon_o E$ and $\mu_o H$ have significance in electromagnetic theory as displacement density and magnetic induction respectively, the quantity $\delta_o v$ is identifiable as the density of electron momentum, and $\kappa_o p$ as the electron condensation (i.e., the ratio of the perturbation of electron concentration to its time average).

For the degenerate case of no static magnetic field, it is possible to separate Φ into a transverse field Φ_T and a longitudinal field Φ_L, which are respectively

$$\Phi_T = \begin{vmatrix} E_T \\ H \\ v_T \end{vmatrix} \quad , \quad \Phi_L = \begin{vmatrix} E_L \\ v_L \\ p \end{vmatrix} \tag{10}$$

Similarly Ψ separates into a source Ψ_T which contributes only to transverse fields, and Ψ_L which contributes only to longitudinal fields, i.e.

$$\Psi_T = \begin{vmatrix} J_T \\ K \\ -F_T \end{vmatrix} \quad , \quad \Psi_L = \begin{vmatrix} J_L \\ -F_L \\ -Q \end{vmatrix} \tag{11}$$

A discussion of the separation of J into transverse and longitudinal parts has been given by Balmain [1963]; the separation of F follows a similar argument. A useful continuity equation for the source is

$$\nabla \cdot J \equiv \nabla \cdot J_L = -i\omega\rho + \frac{\delta_o e}{m} Q \tag{12}$$

where ρ is the source charge density; this implies that ρ is a derived quantity. The field equations may be expressed

$$\begin{aligned} \Psi_T &= M_T \; \Phi_T \\ \Psi_L &= M_L \; \Phi_L \end{aligned} \tag{13}$$

where

$$M_T = \begin{vmatrix} -i\omega\varepsilon_o & \nabla\wedge & \frac{\delta_o e}{m} \\ -\nabla\wedge & -i\omega\mu_o & 0 \\ -\frac{\delta_o e}{m} & 0 & -i\omega\delta_o U \end{vmatrix}$$

$$M_L = \begin{vmatrix} -i\omega\varepsilon_o & \frac{\delta_o e}{m} & 0 \\ -\frac{\delta_o e}{m} & -i\omega\delta_o U & -\nabla \\ 0 & -\nabla\cdot & -i\omega\kappa_o \end{vmatrix} \tag{14}$$

Thus, it may be noted that the transverse field equation is exactly that for a cold isotropic plasma. The nature of the longitudinal field equation may be more fully understood if E_L and v_L are identified as

$$E_L = -\nabla\xi$$
$$v_L = -\nabla\eta \tag{15}$$

where ξ and η are scalar potential functions. In addition, an operator, D_L, which gives a diagonal form to $D_L M_L \Phi_L$ is

$$D_L = \begin{vmatrix} \frac{1}{i\omega\varepsilon_o}(\nabla^2+k_p{}^2U)\nabla\cdot & \frac{\delta_o\kappa_o e}{\varepsilon_o m}\nabla\cdot & \frac{\delta_o e}{i\omega\varepsilon_o m}\nabla^2 \\ \frac{\kappa_o\delta_o e}{\varepsilon_o m}\nabla\cdot & i\omega\kappa_o\nabla\cdot & \nabla^2 \\ \frac{\delta_o e}{i\omega\varepsilon_o m}\nabla\cdot & \nabla\cdot & i\omega\delta_o U(1+\chi_o) \end{vmatrix} \tag{16}$$

where χ_o is the isotropic cold plasma susceptibility ($\chi_o = -\omega_p{}^2/\omega^2 U$), and k_p is the plasma wave number given by

$$k_p = \omega_p\sqrt{\delta_o\kappa_o} = \frac{k_D}{\sqrt{\gamma}} \tag{17}$$

where k_D is the Debye wave number. Operation on both sides of the longitudinal field equation with D_L results in

$$D_L\Psi_L = (\nabla^2 - \alpha^2)\begin{vmatrix} \nabla^2 & 0 & 0 \\ 0 & \nabla^2 & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} \xi \\ \eta \\ p \end{vmatrix} \tag{18}$$

where

$$\alpha^2 = k_p{}^2\left(\frac{1+\chi_o}{\chi_o}\right) \tag{19}$$

It is evident from the form of equation (18) that the scalar fields ξ

η and p may propagate in a source free region with complex wave number $-i\alpha$; this represents the electroacoustic field. In addition ξ and η may each satisfy Poisson's equation, which give the electrostatic electric and velocity fields. Due to the form of $D_L \Psi_L$ it can be further ascertained that the electrostatic fields are those which would exist in a cold plasma.

In electromagnetic theory a useful concept is that of reaction [cf. Rumsey, 1954]. To extend this to the present case it is helpful to introduce the field $\tilde{\Phi}$, which is defined as

$$\tilde{\Phi} = \begin{vmatrix} E \\ -H \\ -v \\ p \end{vmatrix} \tag{20}$$

Then for two sources Ψ_A and Ψ_B, the Lorentz relation is

$$\tilde{\Phi}_A^T \, M \, \Phi_B - \tilde{\Phi}_B^T \, M \, \Phi_A = \tilde{\Phi}_A^T \, \Psi_B - \tilde{\Phi}_B^T \, \Psi_A \tag{21}$$

For the special case of no magnetic field the left-hand side of this equation may be manipulated into the form

$$\nabla \cdot (E_B \wedge H_A - E_A \wedge H_B + v_A p_B - v_B p_A)$$

Upon integration of both sides of equation (21) over a sufficiently large volume, with the assumption of some loss, the left-hand side vanishes, and the resulting reciprocity, or reaction, relation is

$$\int_A d^3r \; \tilde{\Phi}_B{}^T \Psi_A = \langle AB \rangle = \int_B d^3r \; \tilde{\Phi}_A{}^T \; \Psi_B \tag{22}$$

where each integration is over the support of the appropriate source. Rumsey [1954] has designated the quantity <AB> the reaction of the sources A and B. Its physical significance is that with field and source terms defined for unit current input to the terminals of each source, -<AB> is the mutual impedance, while for unit voltage input it is the mutual admittance. In the limiting case where Ψ_A and Ψ_B are the same source, the self reaction leads to the input impedance or admittance. These observations are identical to those in network theory based on the reciprocity theorem.

3. An Approximate Sheath Boundary Condition

The sheath which surrounds a conducting body in a plasma may be characterized, insofar as the Lorentz gas assumptions are concerned, as a region in which the electron concentration is much less than the ambient concentration in the plasma. The sheath boundary is not abrupt, but the major portion of the change in concentration may take place in about one Debye length [cf. Pavkovich, 1964]. Thus for fields with wavelengths much greater than a Debye length, it is permissible to approximate a sheath as an abrupt discontinuity in electron concentration. Fields which propagate with wavelengths nearly equal to the Debye length in the hydrodynamic theory are physically unrealizable, due to Landau damping. Thus the ranges of applicability of the present sheath approximation and of the hydrodynamic theory are the same.

The existence of a sheath about a body in a plasma is a natural thing which occurs due to the thermal motion of electrons and ions, which results in a static negative charge on the body and a surrounding region of excess positive ions. However, a conjecture of an abrupt discontinuity in electron concentration at the sheath edge has the physical implication that an electric double layer exists as a barrier. For such a model to have physical significance it is necessary that the surface charge on the inner shell of the assumed electric double layer be equal to that on the body in the actual case, and that on the outer shell be equal to the total excess positive ion charge in the sheath. For equilibrium conditions it is evident that the inner and outer charges are equal in magnitude but of opposite sign.

To elucidate the similarity of the present barrier approximation to an actual sheath consider a one-dimensional situation of a plane conductor in a plasma. The actual conductor is presumed to have static surface charge σ per unit area (a negative quantity), the condensation of positive charge in the sheath is S_+, and the ambient positive ion concentration in the plasma outside the sheath is assumed to equal that of the electrons, δ_o/m. Thus the distribution of charge in the actual sheath is

$$\sigma\delta(x) + \frac{\delta_o e}{m} S_+(x)$$

where $\delta(x)$ is the Dirac delta function, and the x direction is taken to be normal to the conductor surface. The static electric field, E_{DC}, for $x > 0$ is

$$E_{DC}(x) = \hat{x}\,\frac{1}{\varepsilon_o}\left[\sigma + \frac{\delta_o e}{m}\int_o^x d\zeta\; S_+(\zeta)\right] \tag{23}$$

where $\hat{x}$ is a unit vector in the x direction and ζ is a dummy variable. From this, the electric potential of the conductor may be found to be

$$V_{DC}(0) = \hat{x}\cdot\int_o^\infty dx\; E_{DC}(x) = \frac{\sigma}{\varepsilon_o} \tag{24}$$

Hence an electron which enters the sheath from the plasma and reaches the conductor looses kinetic energy $-\sigma e/\varepsilon_o$. For the electric double layer barrier approximation, the static charge distribution model is

$$\sigma\delta'(x-x_o)$$

where $\delta'(x-x_o)$ is the derivative of the Dirac delta function, and x_o is the coordinate of the barrier. The static electric field for this case is

$$E_{DC} = \hat{x}\frac{\sigma}{\varepsilon_o}\,\delta(x-x_o) \tag{25}$$

and the electric potential as a function of x is

$$V_{DC} = \int_x^\infty E_{DC}(\zeta)d\zeta = \frac{\sigma}{\varepsilon_o}\,[\Theta(x) - \Theta(x-x_o)] \tag{26}$$

where Θ is the Heaviside function (or unit step function), i.e.

$$\Theta(x-x_o) = \int_{-\infty}^{x} dx\delta(x-x_o) \tag{27}$$

Therefore the kinetic energy lost by an electron in traversing the approximate sheath boundary, $-\sigma e/\varepsilon_o$, is identical to that lost by an electron in traversing the actual sheath. In either case electrons not possessing the required initial energy are reflected, and in either case the effect of electron molecule collisions within the sheath is to increase the apparent barrier potential. In the ionosphere it is not uncommon for the electron mean free path length to greatly exceed the Debye length, and thus the neglect of collisions within the sheath should not yield misleading conclusions.

Electrons which traverse the sheath and reach the conducting plane must have initial speeds in the -x direction which exceed $\sqrt{-2\sigma e/\varepsilon_o m}$. Thus the electron current density through the sheath is

$$\vec{J}_o = \hat{x}e\int_{-\infty}^{\infty} dv_z \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{-\sqrt{-\frac{2\sigma e}{\varepsilon_o m}}} dv_x \; f(v) \; v_x \tag{28}$$

where f(v) is the electron velocity distribution function outside the sheath, and again collisions within the sheath are neglected. Assuming f(v) to be Maxwellian, $\vec{J}_o$ becomes

$$\vec{J}_o = \hat{x}\frac{\delta_o e}{m}\sqrt{\frac{k_B T}{2\pi m}} \;\; \exp\left(\frac{\sigma e}{\varepsilon_o k_B T}\right) \tag{29}$$

It should be noted that this result is applicable for negative values of σ. The mathematical ramification of positive σ is that the exponential term must be replaced by unity; and the resulting boundary condition is that characterized as absorptive by Balmain [1966].

The presence of a time varying longitudinal field in the plasma may cause fluctuation of the electron condensation at the sheath boundary. Because expression (29) depends both on electron concentration and temperature, it is evident that if these quantities were to fluctuate at the sheath boundary a similar fluctuation would be impressed upon $\vec{J}_o$. Thus the current density through the sheath at frequency ω may be approximated for small condensations by

$$\mathcal{J}(\omega) \cong -\Delta s(\omega,x_o)\,\frac{d\mathcal{J}_o}{ds} \tag{30}$$

where $\Delta s(\omega,x_o)$ is the discontinuous change in the time-varying electron condensation across the sheath barrier. Performing the differentiation for adiabatic conditions gives

$$\mathcal{J}(\omega) = -\hat{x}\Delta s(\omega,x_o)\Gamma \tag{31}$$

where Γ is the current density coefficient

$$\Gamma = [1 + (\gamma-1)\,(\frac{1}{2} - \frac{\sigma e}{\varepsilon_o k_B T})]\,\mathcal{J}_o\cdot\hat{x} \tag{32}$$

If it is assumed that there is no time-varying concentration of surface charge on the sheath barrier (at $x = x_o$), then the density of current through the sheath is also given by

$$\mathcal{J}(\omega) = -\frac{\delta_o e}{m}\,v(x_o+\Delta x) = -\frac{\delta_o' e}{m}\,v'(x_o-\Delta x) \tag{33}$$

where the prime denotes quantities within the sheath, and Δx is an arbitrarily small positive displacement. The equivalence of equations (31) and (33) is the essence of the electron current boundary condition for the abruptly discontinuous electron concentration model of the sheath.

4. Application to Antenna Impedance Problem

To further elucidate the utility of the present hydrodynamic approach, it will be useful to apply it to the study of the electrical behavior of an antenna in a warm isotropic plasma. For this purpose a relatively simple geometry, similar to the Hertzian dipole, will be employed. A sketch of the assumed antenna is shown in Figure 1. It consists of two spheres, each of radius R, which are separated by the distance L, where the dimensions are assumed to satisfy the condition

$$\frac{1}{k_o} >> L >> R, \ 1/k_p \tag{34}$$

and where k_o is the free space electromagnetic wave number ($=\omega/c$), and k_p is the plasma wave number ($=\omega_p/a$). Connecting the spheres is a cylindrical conductor, on the surface of which is essentially a divergenceless surface current, equal in total to I_g, the generator current. At the surface of each sphere the current density is presumed to have a divergence, leading to an approximately uniform oscillating surface charge on each sphere. Thus for the assumed geometry J_T exists as a uniform line current between the spheres while J_L has divergence only at the surfaces of the spheres.

The transverse field, Φ_T, is a solution to the equation $\Psi_T = M_T\Phi_T$, as was indicated in equation (13). In the present case Ψ_T consists only of the current J_T, and thus the transverse field is approximately that radiated by a Hertzian dipole of moment LI_g in a cold plasma of susceptibility χ_o. Deschamps [1962] has shown that the impedance of an antenna in a conducting medium having complex dielectric constant is related to its free space impedance by

$$Z(\omega, \varepsilon) = \frac{1}{n} Z(n\omega, \varepsilon_o) \tag{35}$$

where n is the refractive index of the conducting medium (i.e., $\varepsilon = n^2\varepsilon_o$). In free space the current element of a Hertzian dipole contributes, to a first order approximation, only to the radiated fields, while the induction and electrostatic fields are due to the divergence of the current at either end of the dipole [cf. Jordan, 1950]. Thus the contribution of J_T to the dipole impedance in free space is the radiation resistance ($\cong 20\, k_o^2 L^2$). Applying relation (35) to the plasma case gives the transverse field contribution to the antenna impedance to be

$$Z_T = 20\, k_o^2\, L^2 \sqrt{1+\chi_o} \tag{36}$$

The effect of a sheath about J_T is negligible in this approximation, just as the radius of the conductor is unimportant.

To explain the effects of the electrostatic and electroacoustic (i.e., longitudinal) fields on antenna behavior, it will be helpful to first examine the fields of a single sphere with oscillating surface charge, and later to compute the mutual effect of two such spheres on antenna impedance.

The general form of solution to equation (18) in a source free region in spherical coordinates is

$$\begin{vmatrix} \xi \\ \eta \\ p \end{vmatrix} = \frac{1}{r} \begin{vmatrix} A_1 + A_2 e^{-\alpha r} \\ B_1 + B_2 e^{-\alpha r} \\ C e^{-\alpha r} \end{vmatrix} \qquad (37)$$

where r is the radial coordinate, α is defined in equation (19), and A_1, A_2, B_1, B_2, and C are constants. Applying the definitions of ξ and η as given by equations (15) gives the general form of Φ_L to be

$$\Phi_L = \begin{vmatrix} E_L \\ v_L \\ p \end{vmatrix} = \begin{vmatrix} \frac{\hat{r}}{r^2} [A_1 + A_2(\alpha r+1)e^{-\alpha r}] \\ \frac{\hat{r}}{r^2} [B_1 + B_2(\alpha r+1)e^{-\alpha r}] \\ \frac{C}{r} e^{-\alpha r} \end{vmatrix} \qquad (38)$$

where $\hat{r}$ is a unit vector in the radial coordinate direction. Assuming that if F_L and Q exist, these act only as surface sources, the form of Φ_L is unchanged in a region where the source current J_L has zero divergence, because of the form of $D_L\Psi_L$; cf. equation (18). The proper form of J_L which has divergence, and hence charge accumulation, only on the spherical conductor is, from equation (12),

$$J_L = -\hat{r} \frac{I_g}{4\pi r^2} \Theta (r-R) \qquad (39)$$

where Θ is the unit step (or Heaviside) function defined previously in equation (27). While this is not a physically realistic current distribution it satisfies the condition that $\nabla \wedge J_L = 0$, and in addition has the necessary form for uniform charge distribution on the sphere. Substitution of expressions (38) and (39) into the longitudinal field equation $\psi_L = M_L \Phi_L$ gives the relationships

$$A_1 = \frac{I_g}{i\omega\varepsilon_o 4\pi} \left(\frac{1}{1+\chi_o}\right)$$

$$B_1 = -\frac{mI_g}{4\pi\delta_o e} \tag{40}$$

$$A_2 = \frac{\delta_o e}{i\omega\varepsilon_o m} \qquad B_2 = \frac{m}{\delta_o e}\left(\frac{\chi_o}{1+\chi_o}\right) C$$

Thus the only unknown quantity is the constant A_2, which must be obtained by application of a boundary condition.

In the following, quantities related to the sheath are designated by the addition of a prime: the sheath radius is R', the susceptibility within the sheath χ_o', etc. It is assumed that the plasma frequency within the sheath is small as compared with the applied frequency, so that the electroacoustic field within the sheath region may be neglected. Thus with the aid of equations (40), Φ_L may be expressed in the form

$$\Phi_L = \begin{vmatrix} \dfrac{\hat{r}\, I_g}{i\omega\varepsilon_o 4\pi r^2}\left(\dfrac{1}{1+\chi_o{}'}\right) \\ -\dfrac{\hat{r}\, m\, I_g}{\delta_o' e 4\pi r^2}\left(\dfrac{\chi_o{}'}{1+\chi_o{}'}\right) \\ 0 \end{vmatrix} [\Theta(r-R) - \Theta(r-R')]$$

$$+ \begin{vmatrix} \dfrac{\hat{r}}{r^2}\left[\dfrac{I_g}{i\omega\varepsilon_o 4\pi}\left(\dfrac{1}{1+\chi_o}\right) + A_2(\alpha r+1)e^{-\alpha r}\right] \\ \dfrac{\hat{r}}{r^2}\left[-\dfrac{m\, I_g}{\delta_o e 4\pi}\left(\dfrac{\chi_o}{1+\chi_o}\right) + \dfrac{i\omega\varepsilon_o m}{\delta_o e} A_2(\alpha r+1)e^{-\alpha r}\right] \\ \dfrac{\delta_o e}{rm}\left(\dfrac{1+\chi_o}{\chi_o}\right) A_2\, e^{-\alpha r} \end{vmatrix} \Theta(r-R') \tag{41}$$

where the unknowns $\chi_o{}'$ and A_2 must be evaluated by satisfying the boundary condition at $r = R'$ that electron momentum is conserved. As given by equation (33), this may be expressed in terms of the coefficients of equation (41) as

$$\hat{r}\cdot\mathcal{J}(\omega) = \frac{I_g}{4\pi R'^2}\left(\frac{\chi_o{}'}{1+\chi_o{}'}\right) = \frac{I_g}{4\pi R'^2}\left(\frac{\chi_o}{1+\chi_o}\right) - \frac{i\omega\varepsilon_o A_2}{R'^2}(\alpha R'+1)e^{-\alpha R'} \tag{42}$$

Identifying $-\Delta s$ as $\kappa_o p(R')$, equation (31) is equivalent to

$$\hat{r}\cdot\mathcal{J}(\omega) = \frac{\kappa_o\, \delta_o e}{R'm}\left(\frac{1+\chi_o}{\chi_o}\right) A_2 e^{-\alpha R'}\Gamma \tag{43}$$

By equating expressions (42) and (43), the relations

$$A_2 = \frac{I_g e^{\alpha R'}}{i\omega\varepsilon_o 4\pi}\left(\frac{\chi_o}{1+\chi_o}\right)\left(\frac{1}{\alpha R'+1+\frac{\kappa_o\delta_o eR'}{i\omega\varepsilon_o m}\left(\frac{1+\chi_o}{\chi_o}\right)\Gamma}\right) \tag{44}$$

and

$$\chi_o' = -\left\{1+\frac{i\omega\varepsilon_o m}{\kappa_o\delta_o eR'\Gamma}\left[\alpha R'+1+\frac{\kappa_o\delta_o eR'}{i\omega\varepsilon_o m}\left(\frac{1+\chi_o}{\chi_o}\right)\Gamma\right]\right\}^{-1} \tag{45}$$

are obtained. These may be substituted into equation (41) to complete the formulation of Φ_L for an isolated sphere.

To determine the impedance of the assumed antenna it is necessary to consider the contributions of both the transverse and longitudinal fields. Examination of the configuration, shown in Figure 1, will indicate that the impedance due to these fields are in effect additive at the antenna terminals. Thus the input impedanze Z is the sum

$$Z = Z_T + Z_L \tag{46}$$

where Z_T is due to transverse field, and is given by equation (36), while Z_L, that due to the longitudinal field, must be determined.

The latter may be accomplished by application of equation (22) to the self reaction, i.e.

$$Z_L = -\int d^3r\ \tilde{\Phi}_A^T\ \Psi_A \tag{47}$$

with the elements of $\tilde{\Phi}_A^T$ equal to the appropriate quantities of Φ_L. In the present formulation the only term in the source matrix, Ψ_A, is J_T, which is essentially a line current of unit amplitude ($I_g = 1$) between the two spheres. Because of the symmetry which exists in the assumed configuration the field of each sphere contributes identically to Z_L, and therefore equation (47) may be expressed

$$Z_L = 2\int_R^{L-R} dr\,\hat{r}\cdot E_L\,(I_g = 1) \tag{48}$$

where the field E_L is that given by equation (41) with $I_g = 1$. Performing the indicated integration gives

$$\begin{aligned} Z_L &= \frac{2}{i\omega\varepsilon_o 4\pi R}\left\{\frac{R'-R}{R'(1+\chi_o')} + \frac{R(L-R')}{LR'(1+\chi_o)}\right\} \\ &+ 2\,\frac{A_2}{I_g}\left\{\frac{e^{-\alpha R'}}{R'} - \frac{e^{-\alpha(L-R)}}{L-R}\right\} \end{aligned} \tag{49}$$

where A_2 and χ_o' are given by equations (44) and (45) respectively. Note that the first part of Z_L represents the electrostatic impedance of the antenna in a cold plasma, and that it approaches the appropriate form as χ_o and χ_o' approach zero (i.e., free space conditions). The second part of Z_L is due to the electroacoustic fields; it indicates that at frequencies at which α is essentially imaginary ($\omega>\omega_p$) there is an oscillating dependence of Z_L on the dipole length L. This is because L may be several electroacoustic wavelengths, and the resulting behavior is not unlike that of an antenna in free space which is very long in terms of the electromagnetic wavelength. It may also be observed that Z_L reduces to twice the impedance of an isolated sphere if L is allowed to become infinite, and furthermore that if the sheath is collapsed ($R' = R$) and the surface barrier potential made infinite ($-\sigma e/\varepsilon_o \to \infty$) then this result is exactly that derived by Fejer [1964] for the perfectly reflecting boundary condition. In addition, expression (49) becomes exactly twice the impedance of an isolated sphere as calculated by Balmain [1966] for an absorptive boundary condition if $L \to \infty$, $R' = R$, $\sigma e/\varepsilon_o = 0$ (i.e., no sheath barrier), and Balmain's absorption coefficient, a, is set equal to

$$a = \frac{1+\gamma}{2}\sqrt{\frac{\gamma}{2}} \tag{50}$$

For a nominal value of γ, say $\gamma = 2$, this is approximately equal to Balmain's limiting value of $a = \sqrt{2/\pi}$. Thus the present theory is sufficiently general to include the results found by other workers for these limiting cases.

The impedance of the assumed antenna, the sum of Z_T and Z_L, may now be expressed, in terms of electromagnetic, electrostatic, and electroacoustic contributions, as

$$\begin{aligned} Z = {} & 20\, k_o^2\, L^2 \sqrt{1+\chi_o} && \text{(electromagnetic)} \\ & + \frac{2}{i\omega\varepsilon_o 4\pi R}\left\{\frac{R'-R}{R'(1+\chi_o')} + \frac{R(L-R')}{LR'(1+\chi_o)}\right\} && \text{(electrostatic)} \\ & + 2\,\frac{A_2}{I_g}\left\{\frac{e^{-\alpha R'}}{R'} - \frac{e^{-\alpha(L-R)}}{L-R}\right\} && \text{(electroacoustic)} \end{aligned} \tag{51}$$

Because of condition (34) it is obvious that the electromagnetic term is relatively unimportant, as is the case for any small antenna.

5. The Direct Current Collected by a Probe

A resonant behavior of the direct current collected by a probe in a warm plasma has been observed as the applied frequency is swept through a range which includes the plasma frequency [cf. Takayama et al., 1960 or Miyazaki et al., 1960]. Fejer [1964] has pointed out that for a reflecting boundary condition (corresponding to an infinite potential barrier in the present analysis) the electron condensation at the sheath exhibits a similar resonant behavior at a frequency which is notably less than the plasma frequency, and which depends upon the dimensions of the probe as well as upon the plasma parameters. With a finite potential barrier supporting the sheath the effect of the electron condensation at the sheath boundary is to influence the transport of electrons through the sheath.

The current density through the sheath was treated linearly in the previous discussion. A more accurate representation would be to expand the current as a Tayor series in powers of the electron condensation, and obtain the direct current as the time average. Thus by accounting for the second order nonlinearity in equation (29) with adiabatic conditions, the direct current to the probe is

$$I_{DC} \cong - \frac{4\pi R'\delta_o e}{m} \sqrt{\frac{k_B T}{2\pi m}}\; e^{-\theta} \qquad (52)$$

$$\cdot \left\{ 1 + \frac{1}{2} \left| s(R') \right|^2 \left[(\gamma-1)\left(\frac{1}{2}+\theta\right) + \frac{(\gamma-1)^2}{2} \left(\theta+\theta^{\,2}\right) \right] \right\}$$

where s(R') is the electron condensation at the barrier

$$s(R') = \kappa_o p(R') \tag{53}$$

and θ represents the barrier potential coefficient

$$\theta = -\frac{\sigma e}{\varepsilon_o k_B T} \tag{54}$$

By inspection of expression (52) it is evident that I_{DC} will have a resonant maximum only if the magnitude of s(R') does likewise. However the previously derived expression for p, in equation (41), does not inherently posess a resonant behavior. Rather the resonance must be due to the behavior of I_g, which is a function of applied frequency, unless the r.f. source has infinite impedance. Thus for a constant voltage r.f. generator the quantity s(R') to be used in expression (52) must be that for $I_g = V_g/Z$, or

$$s_V(R') = \frac{\kappa_o \delta_o e\, V_g}{i\omega\varepsilon_o 4\pi R' m Z} \cdot \left(\frac{1}{\alpha R' + 1 + \frac{\kappa_o \delta_o e R'}{i\omega\varepsilon_o m}\left(\frac{1+\chi_o}{\chi_o}\right)\Gamma} \right) \tag{55}$$

where V_g is the generator voltage (peak) and Z is the impedance given by equation (51). It is evident that, in this analysis, the lineqrized theory is valid only if $|s| << 1$; hence the extent to which equation (52) can represent the dependence of I_{DC} on either ω or V_g is limited by the linearization inherent in the formulation of expression (4).

6. Numerical Calculations

To further elucidate the results of the present analysis, it is useful to resort to numerical calculations. Rather than generalize, the subsequent analysis is specifically related to typical ionospheric experimental conditions; this is done to demonstrate the utility of the present hydrodynamic approach to problem solving, and not necessarily the utility of the assumed antenna configuration as a probe. The following parameter values are chosen as representative of possible ionospheric experimental conditions:

electron concentration (δ_o/m)	10^{11} meter^{-3}
electron temperature (T)	300°K
specific heat ratio (γ)	3
antenna length (L)	.5 meter
sphere radius (R)	.05 meter
sheath thickness (R'-R)	$2/k_p$
r.f. source voltage (V_g)	.1 volt
barrier potential coefficient (θ)	3-5

In the subsequent analysis, the behavior of such an antenna, as given by the present theory, will be illustrated, as will the effect of variation of several of the experimental parameters.

For the aforementioned nominal parameters, Figure 2 shows the behavior of the real and imaginary parts of the antenna admittance, Y, where

$$Y = Z^{-1} \text{ ohm}^{-1}$$

and of the direct current collected by one sphere, I_{DC}, as functions of normalized frequency ω/ω_p for two values of the barrier potential coefficient, $\theta = 3$ and 5. This clearly shows the resonant natures of both Y and I_{DC}, the latter having the general features noted by Takayama et al. [1960] in laboratory experiments. It also indicates a broadening effect upon the resonance with a decrease in the potential barrier coefficient. These curves are in essence replotted in Figure 3, along with those which indicate the antenna behavior for a relatively large collision frequency, $\nu = 10^6$ (i.e. $\nu/\omega_p \cong .05$). It is evident that the collisional effect on Y and I_{DC} is more pronounced for a larger value of θ. For $\nu = 10^4$, however, the deviations from the $\nu = 0$ curves are so small as to be undiscernable on the graphs.

The effect of collisions is most pronounced in the behavior of impedance near $\omega/\omega_p = 1$. Figure 4 shows the resonance-like effect which occurs near the plasma frequency for several values of collision frequency. The dependence of the slope of the lower part of this resonance on collision frequency suggests a possible scheme for its determination; and the occurrence of this resonance almost at the plasma frequency suggests a means for the determination of the latter. However, the possibility of a "stray" shunt capacitance unaccounted for in the analysis is sufficient to make such determinations futile. This futility is illustrated in Figure 5, in which the real part of Z is shown for $\nu = 10^4$ for several values of "stray" shunt capacitance ranging from 0 to 5 pf. The apparent effects of such a perturbation is to shift the resonance away from the plasma frequency, and to broaden the resonance as well.

The sheath thickness, chosen arbitrarily as $2/k_p$, has some effect on the behavior of Y and of I_{DC}. This is shown in Figure 6, in which it is apparent that the admittance resonant frequency increases with increasing sheath thickness. An increase in the resonant frequency also results from a decrease in the sphere radius, as was noted by Fejer [1964] and is illustrated in Figure 7.

In the discussion following equation (49) it was noted that for $\omega/\omega_p > 1$ an oscillating dependence of Z_L on frequency is predicted. This is due to the mutual coupling of the two spheres via electroacoustic waves, which may have wavelengths significantly less than the distance separating the spheres for $\omega/\omega_p > 1$. The effect is most noticeable in the behavior of the real part of admittance, as is illustrated in Figure 8. That this ripple is related to the separation of the spheres may be ascertained from the difference in its period for the lengths L = .2 and .5 meter. An additional property of L is its effect on the frequency of the resonance of Y and I_{DC}, in that decreasing L tends to increase the resonant frequency.

The foregoing analysis is not entirely applicable to the interpretation of ionospheric experiments, because the latter are conducted in an anisotropic medium. It does indicate, however, the predicted nature of some of the dimensionally dependent features of the behavior of an antenna in an isotropic warm plasma, which may reasonably be expected to have counterparts in an anisotropic plasma.

7. Conclusions

The approximate sheath boundary condition, represented by a static electric double layer, has been shown to be capable of representing the static behavior of the sheath about a probe in a warm plasma. With application of the hydrodynamic theory to waves in a warm plasma the boundary condition has been shown to lead to results which are not unlike experimental observations of antenna admittance and direct current collection. Comparison of the present analysis applied to an isolated spherical radiator shows that it leads to the results of other workers for the limiting boundary conditions of perfect reflection and complete absorption. In addition the present analysis includes a useful probe configuration, for possible ionospheric application, which offers the hope of better agreement between theory and experiment because of the clear division of transverse and longitudinal source terms not obtained with an ordinary dipole.

The significance of the present approach to the sheath problem does not lie in its application to the impedance of a sphere in an isotropic warm plasma. Rather the purpose of this paper has been to demonstrate that with an appropriate boundary condition the hydrodynamic theory is capable of giving results which are reasonable and which do not disagree with experimental evidence. A ramification of these results is that the hydrodynamic theory can be applied, with reasonable confidence, to more complicated situations, such as the behavior of an antenna in an anisotropic warm plasma.

8. Acknowledgment

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Figure Captions

Figure 1. Configuration of the assumed antenna.

Figure 2. Admittance and direct current for $\Theta = 3$ and 5 with $\nu = 0$.

Figure 3. Effect of collisions on admittance and direct current for $\Theta = 3$ and 5.

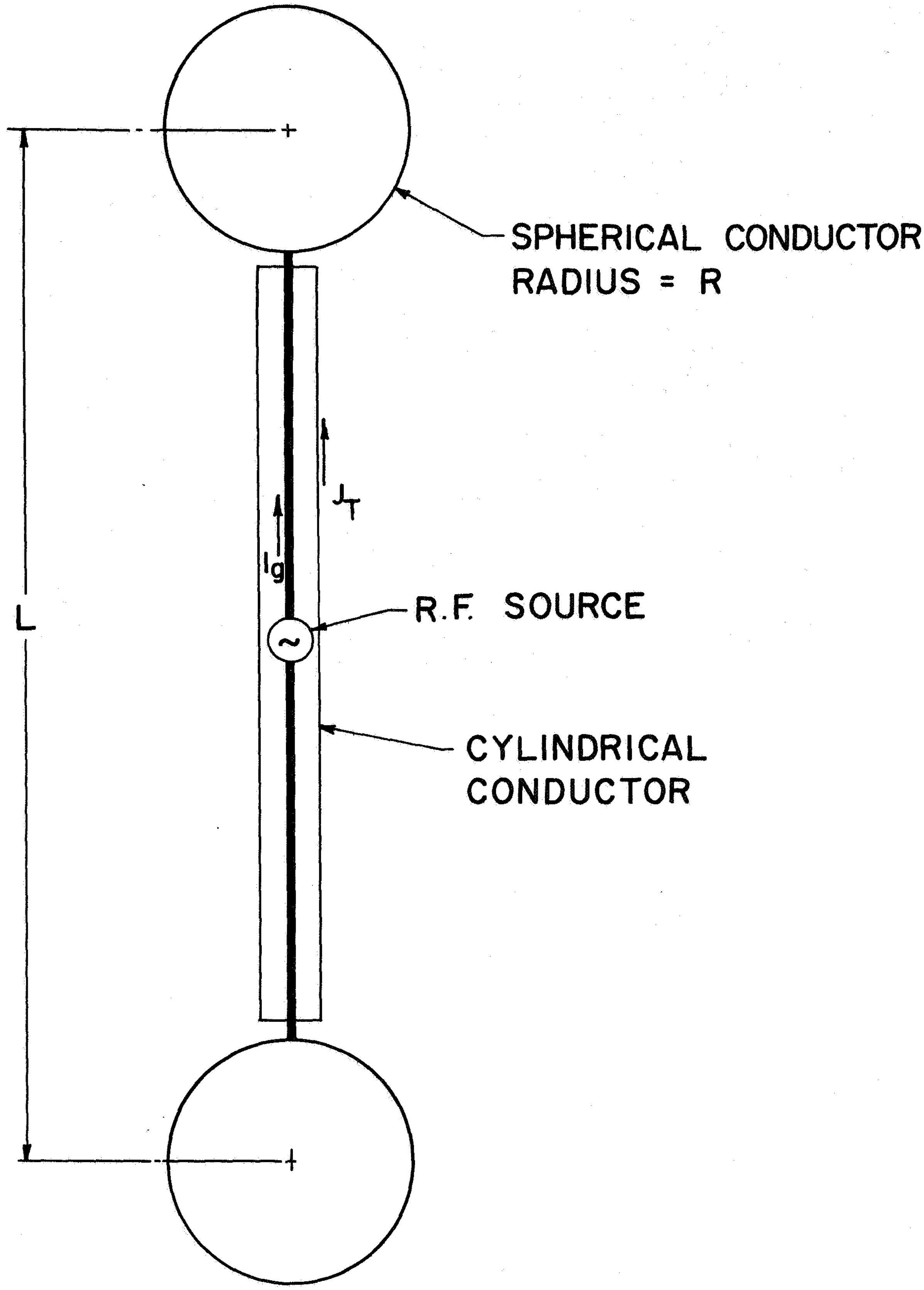
Figure 4. Effect of collisions on impedance near the plasma frequency.

Figure 5. Effect of stray shunt capacitance on impedance near the plasma frequency.

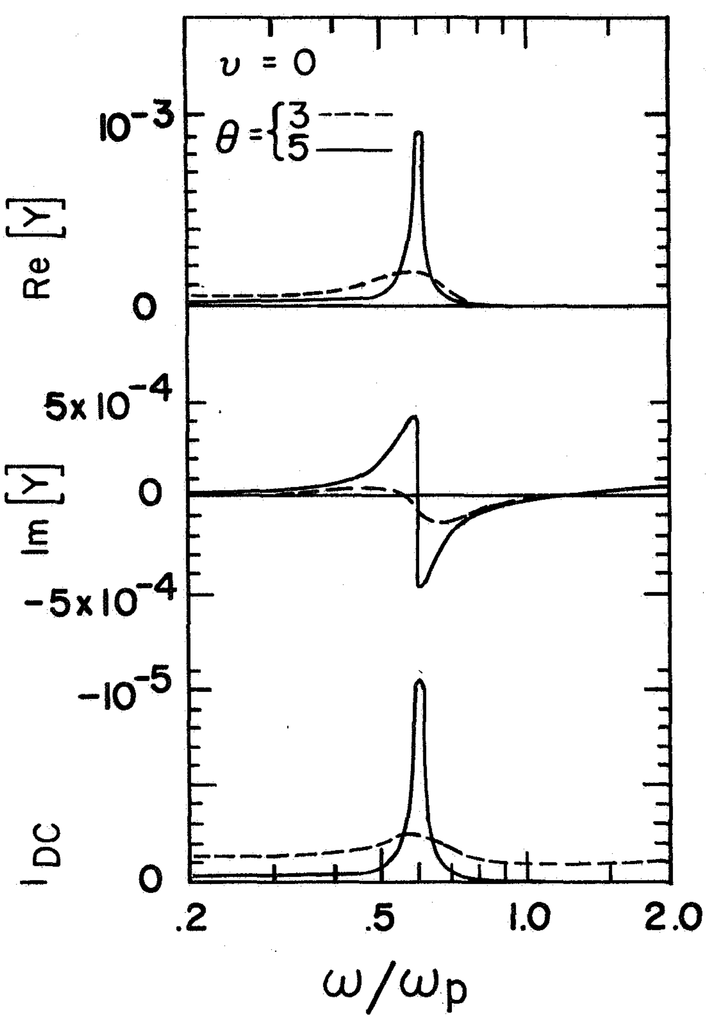
Figure 6. Effect of the assumed sheath thickness on admittance and direct current.

Figure 7. Effect of sphere radius on admittance and direct current.

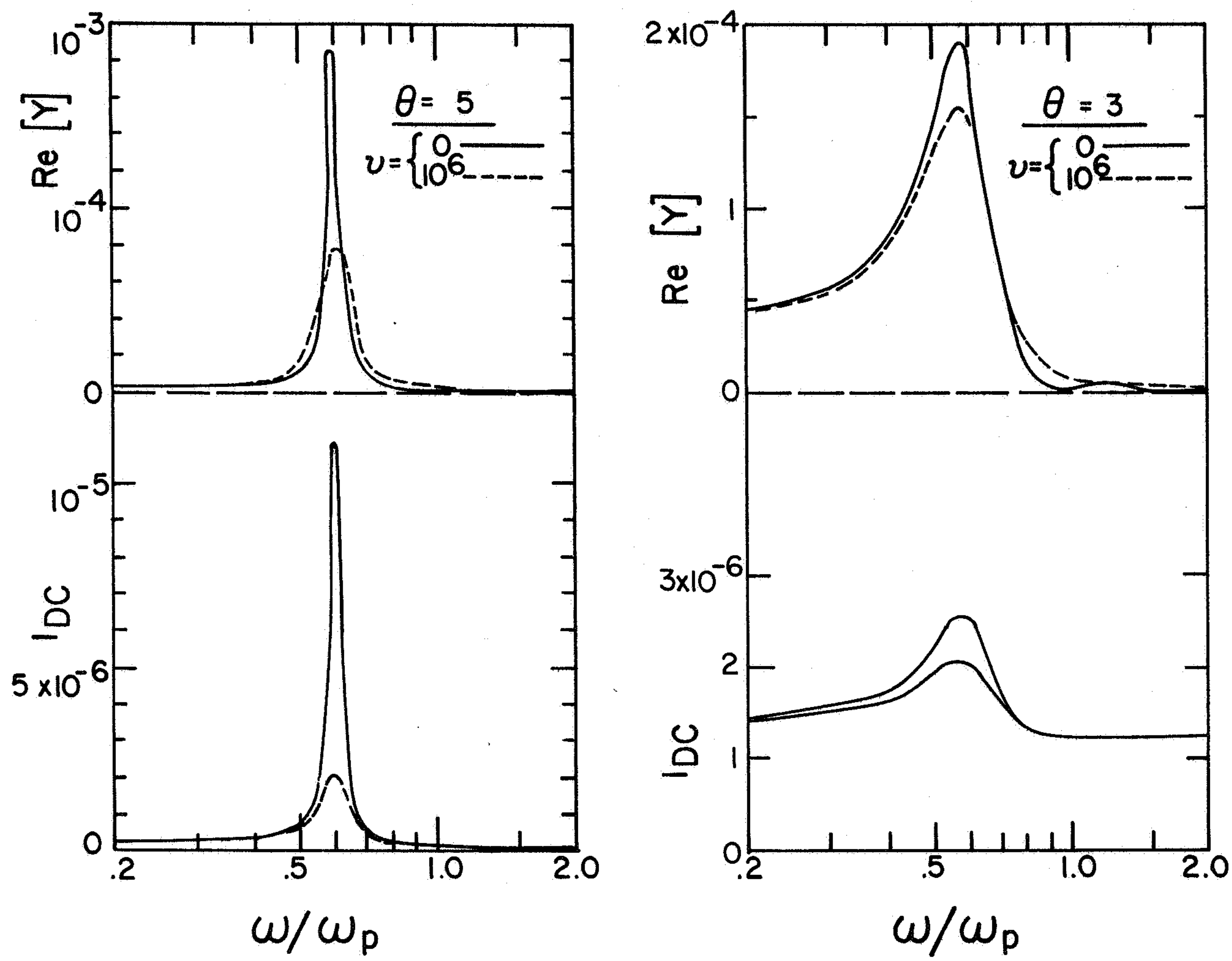
Figure 8. Effect of antenna length on admittance and direct current.



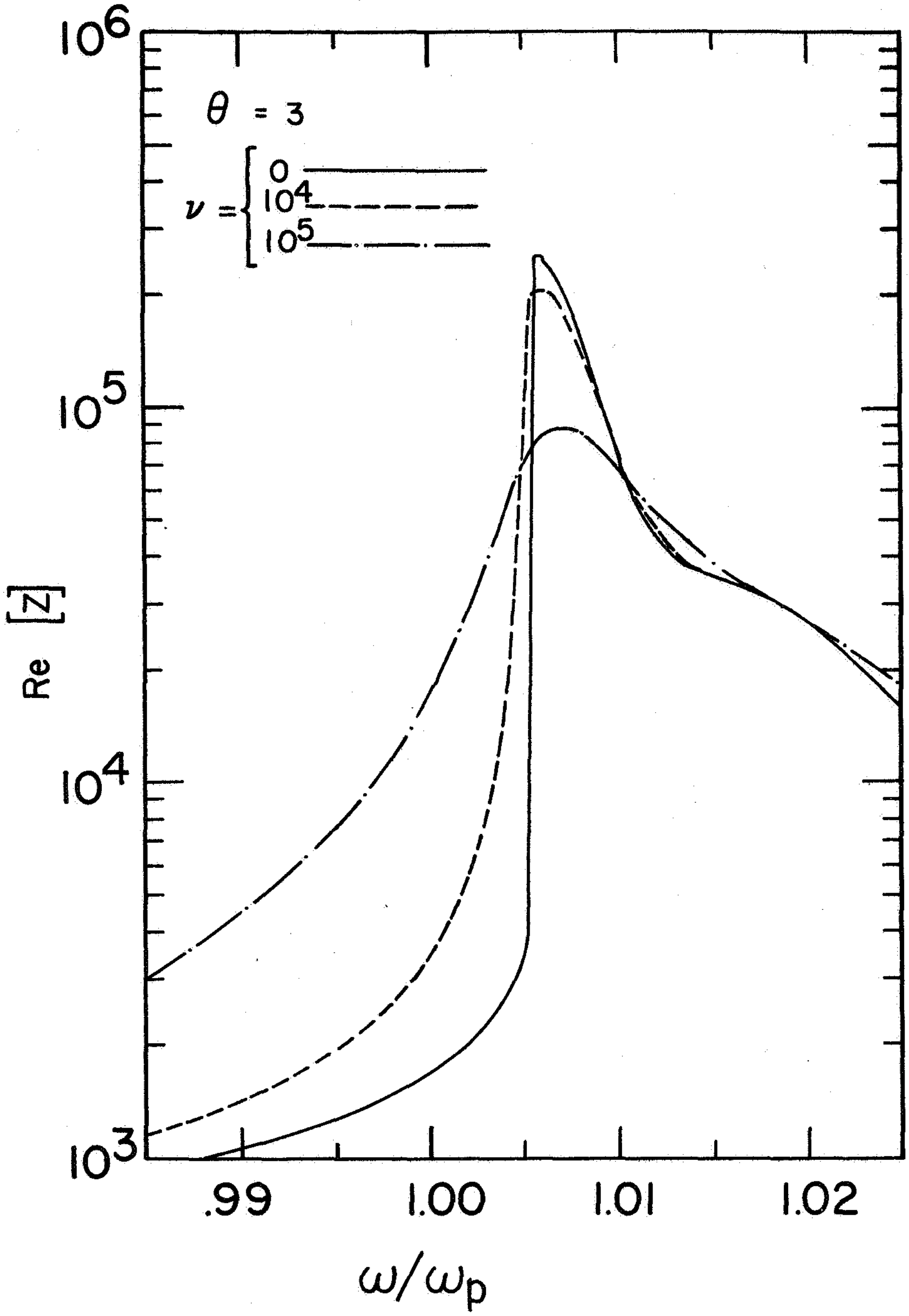
SPHERICAL CONDUCTOR
RADIUS = R
J_T
I_g
L
R.F. SOURCE
CYLINDRICAL
CONDUCTOR



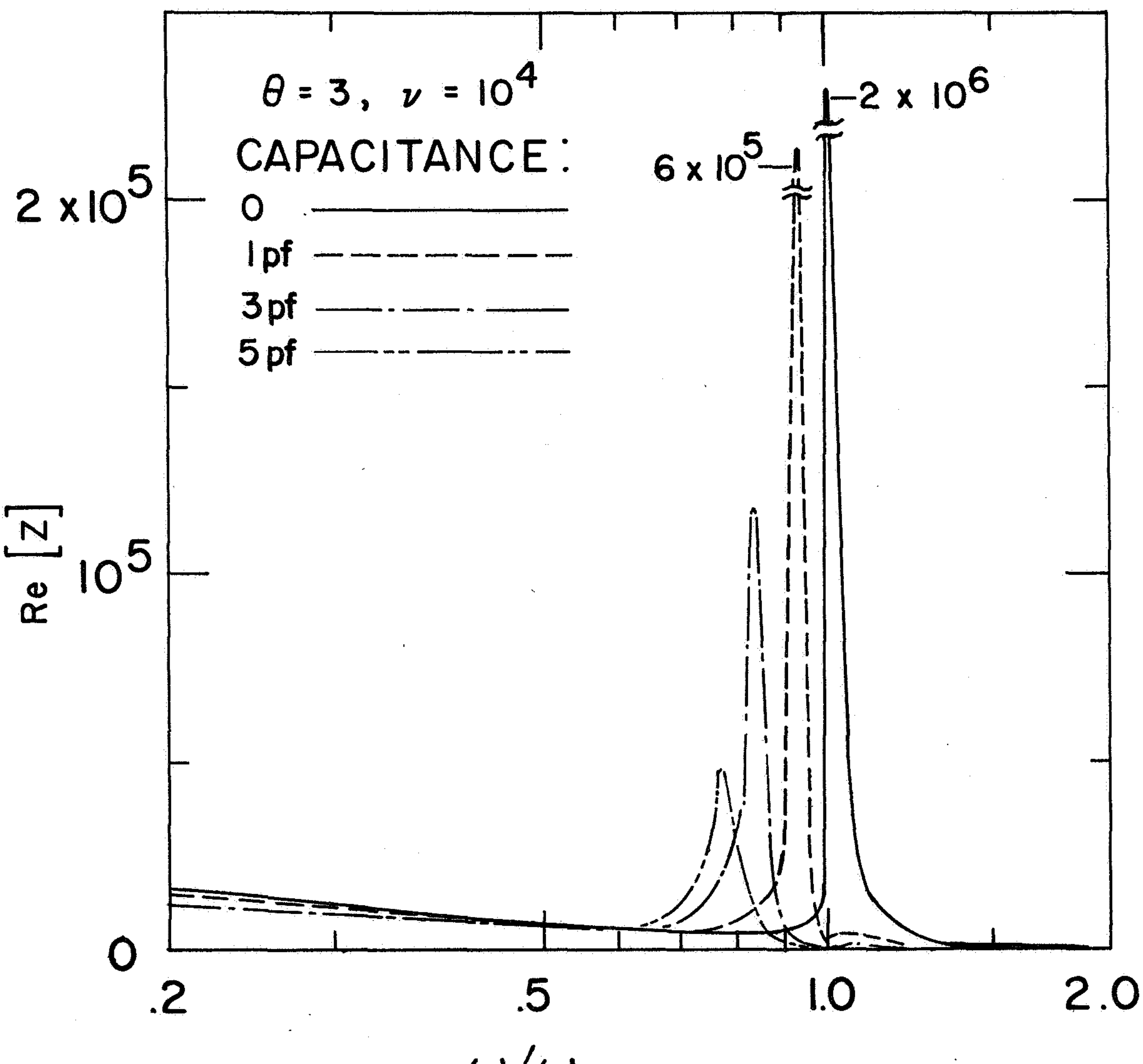
$\upsilon = 0$
$\theta = \{ 3$ ----- 5 —
10^{-3}
Re [Y]
0
5×10^{-4}
Im [Y]
0
-5×10^{-4}
-10^{-5}
I_{DC}
0
.2
.5
1.0
2.0
ω/ω_p



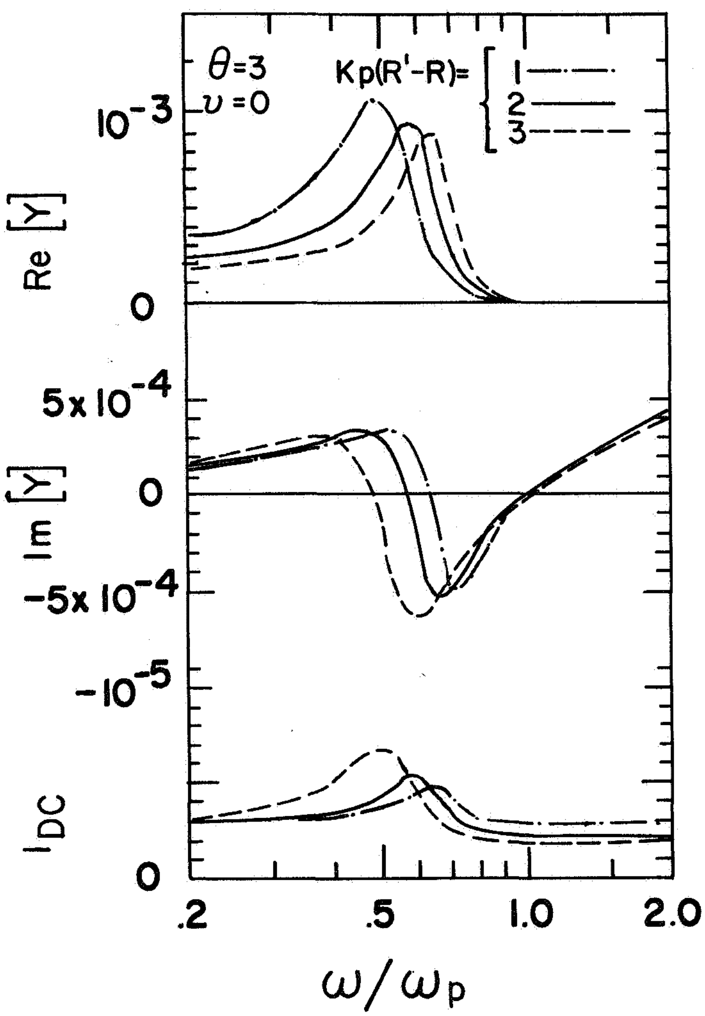
Re [Y]
10^{-3}
10^{-4}
0
$\theta = 5$
$\upsilon = \{ 0, 10^6$
I_{DC}
10^{-5}
5×10^{-6}
0
.2
.5
1.0
2.0
ω/ω_p
2×10^{-4}
1
$\theta = 3$
$\upsilon = \{ 0, 10^6$
3×10^{-6}
2
1
0
ω/ω_p



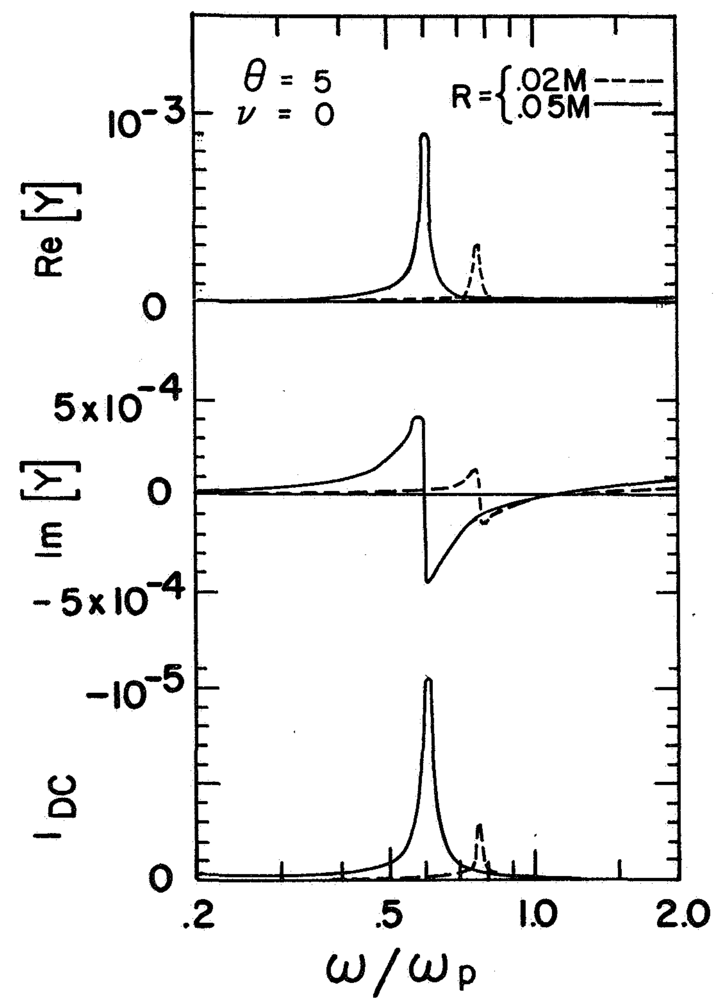
10^6
10^5
10^4
10^3
$\theta = 3$
$\nu =$
0
10^4
10^5
Re [Z]
.99
1.00
1.01
1.02
ω/ω_p



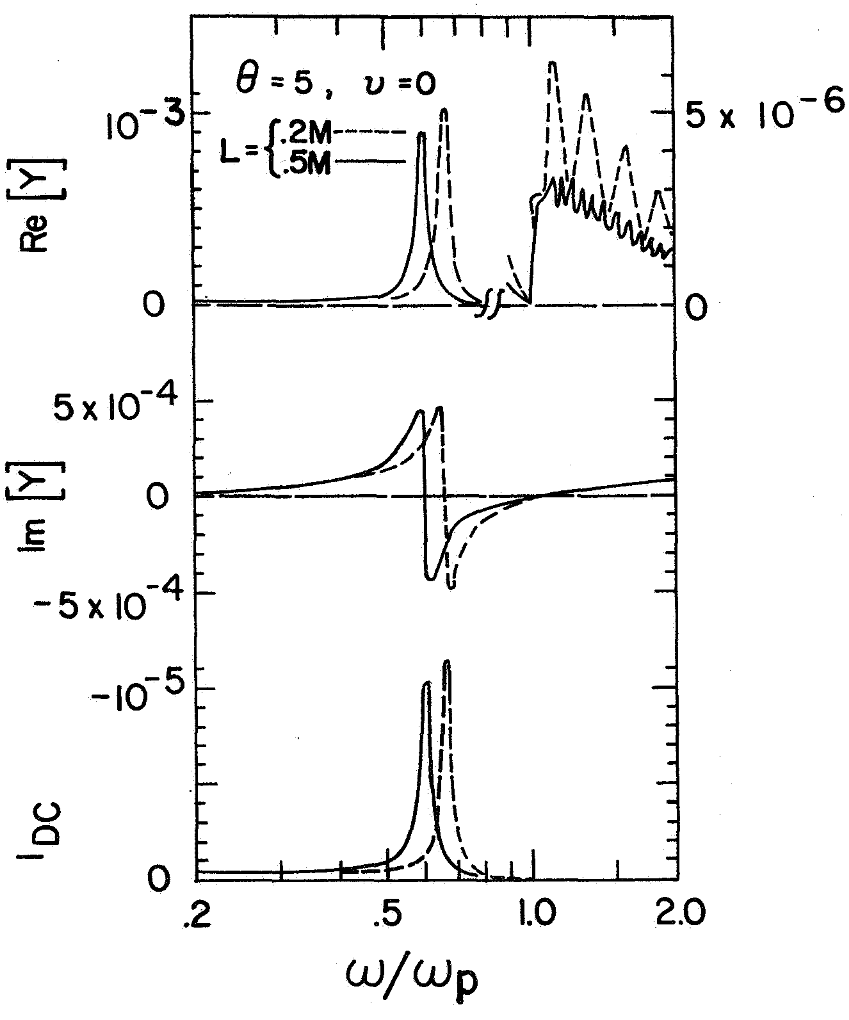
θ = 3, ν = 10^4
CAPACITANCE:
0
1pf
3pf
5pf
2 x 10^6
6 x 10^5
2 x10^5
10^5
0
Re [Z]
.2
.5
1.0
2.0
ω/ωp



θ=3
υ=0
Kp(R'-R)=
1
2
3
10-3
Re [Y]
0
5x10-4
Im [Y]
0
-5x10-4
-10-5
IDC
0
.2
.5
1.0
2.0
ω/ωp



θ = 5
ν = 0
R = { .02M
.05M
10^-3
Re [Y]
0
5 x 10^-4
Im [Y]
0
- 5 x 10^-4
-10^-5
I_DC
0
.2
.5
1.0
2.0
ω/ω_p



θ = 5 , υ = 0
L = {.2M / .5M}
10⁻³
5 x 10⁻⁶
Re [Y]
0
5 x 10⁻⁴
Im [Y]
-5 x 10⁻⁴
-10⁻⁵
I_DC
.2
.5
1.0
2.0
ω/ω_p